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On Some series, Relations Involving Generalised Riemann Zeta Function

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Abstract: In this paper, we establish three general multiple-series identities involving a general class of polynomials and then we deduce some known and new results. Three series identities are also applied to finding the summation formula.

Keywords - *Riemann zeta function, infinite series, summation formula.*

1. Introduction:

The well-known Riemann zeta functions defined as [3].[10]

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} \qquad Re(s) > 1 \tag{1.1}$$

This function can also be defined as [9, p,1]

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \qquad Re(s) > 0; s \neq 1$$

Various interesting generalizations of this function are available in the literature. For example, the generalized(Hurwitz's) zeta function is defined [2,p,24] as

$$\zeta(s,a) = \sum_{n=0}^{\infty} (a+n)^{-s}, \qquad Re(s) > 1; a \neq 0, -1, -2 \tag{1.2}$$

Obviously $\zeta(s,1) = \zeta(s)$ (1.3)

Another interesting generalization of (1.1) and (1.2) is given by [2, p,27, eq.1]

$$\Phi(x, s, a) = \sum_{n=0}^{\infty} (a+n)^{-s} x^n, \quad |x| < 1; a \neq 0, -1, -2 \tag{1.4}$$

The function $\Phi(x, s, a)$ has the following integral expression.

$$\Phi(x, s, a) = \frac{1}{\Gamma s} \int_0^{\infty} t^{s-1} e^{-at} (1 - xe^{-t})^{-1} dt \tag{1.5}$$

Provided that $Re(s) > 1$ and either $|x| \leq 1; x \neq 1$ and $Re(s) > 0$ or $x = 1$ and $Re(s) > 1$.

Obviously $\Phi(1, s, a) = \zeta(s, a)$ (1.6)

And

$$\Phi(1, s, 1) = \zeta(s, 1) = \zeta(s) \tag{1.7}$$

Further if $s = 1$ in(1.4), we have

$$\Phi(x, 1, a) = \sum_{n=0}^{\infty} \frac{1}{(a+n)} x^n$$

$$= \frac{1}{a} 2F_1(1, a; a+1; x), (|x| < 1) \tag{1.8}$$

2. Infinite Series Involving the Function $\Phi(x, s, a)$

An over-two-century-old goldback theorem[4] is

$$\sum_{w \in S} (w-1)^{-1} = 1 \tag{2.1}$$

Where S denotes the set of all nontrivial integer k th powers i.e., $S = \{n^k | n \geq 2, k \geq 2\} = \{4,8,9,16, \dots\}$

In view of the definition of Riemann zeta function (1.1), it has recently been modified as

$$\sum_{k=2}^{\infty} \{\zeta(k) - 1\} = 1 \tag{2.2}$$

Again since

$$1 < \zeta(k) \leq \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi}{6} < 2$$

Therefore

$$0 < \{\zeta(k) - 1\} < 1, k \geq 2$$

so that

$$\{\zeta(k)-1\} = f\{\zeta(k)\} \quad , k \geq 2$$

Thus, the summation formula (1.2) takes the elegent form

$$\sum_{k=2}^{\infty} f\{\zeta(k)\} = 1 \tag{2.3}$$

Where $f(x) = x - [x]$ denotes the fractional part of the real number x.

Related to these summation formulas, several interesting summations have been established by singh and verma[5]. Srivastava [7,8,9] and choi and srivastava[1] etc. Motivated by this work, we establish below certain series involving the function $\Phi(x, \mathrm{s}, a)$.

3. Main Result

(I) $$\Phi(x,\mathrm{s},a).=\left(\frac{x}{1+x}\right)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,n(s)_{n+1}}{(n+2)!}\Phi(x,\mathrm{s}+\mathrm{n}+1,a)$$

$$-\left(\frac{1-x}{1+x}\right)\frac{2}{s-1}\Phi(x,\mathrm{s}-1,a)+\frac{a^{-s}}{1+x}(1+\frac{2a}{s-1}) \tag{3.1}$$

(II) $$\Phi(x,\mathrm{s},a).=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,(s)_{n1}}{(n+1)!}\Phi(x,\mathrm{s}+\mathrm{n},a)-\frac{1}{s-1}\{\left(\frac{1}{x}-1\right)\Phi(x,\mathrm{s}-1,a)-\frac{a^{-s+1}}{x}\} \tag{3.2}$$

Provided that both series are convergent.

Proof of (3,1): Using the well-known binomial expansion and the series form of the $\Phi(x, \mathrm{s}, a)$ defined by (1..4), we readily get

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\Phi(x,\lambda+n,a)t^n = \Phi(x,\lambda,a-t) \quad ,|t|<|a|,\lambda\neq 1 \tag{3.3}$$

Now replace the summation index in (3.3) by n+2 set $\lambda = s - 1$ and divide both the sides of resulting equation by t^2 , we obtain

$$\sum_{n=0}^{\infty} \frac{(s-1)_{n+2}}{(n+2)!} \Phi(x, s+n+1, a) t^n$$

$$= \{\Phi(x, s-1, a-t) - \Phi(x, s-1, a)\} t^{-2} - (s-1)\Phi(x, \mathrm{s}, a) t^{-1} \qquad (3.4)$$

$$0 < |t| < |a|.$$

Differentiating partially both sides of (3.4) with respect to t, and using the following result (which can be established easily)

$$\frac{\partial}{\partial t}\{\Phi(x, s-1, a-t)\} = (s-1)\Phi(x, s, a-t) \quad (3.5)$$

We get

$$\sum_{n=1}^{\infty} \frac{n(s)_{n+1}}{(n+2)!} \Phi(x, \mathrm{s}+\mathrm{n}+1, a) t^{n-1}$$

$$= \{\Phi(x, s, a-t) + \Phi(x, s, a)\} t^{-2} - \frac{2}{s-1}\{\Phi(x, s-1, a-t) - \Phi(x, s-1, a)\} t^{-3}$$

$$0 < |t| < |a|. (3.5)$$

Putting $t = -1$ in (3.5) we find that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n(s)_{n+1}}{(n+2)!} \Phi(x, \mathrm{s}+\mathrm{n}+1, a)$$

$$= \{\Phi(x, \mathrm{s}, a+1).+\Phi(x, \mathrm{s}, a)\} + \frac{2}{s-1}\{\Phi(x, \mathrm{s}-1, a+1) - \Phi(x, \mathrm{s}-1, a) \qquad (3.6)$$

Now using the following modified form of the known result [2,p. 27]

$$x\Phi(x,s,a+1) = \Phi(x,s,a) - a^{-s} \tag{3.7}$$

In (3.6) we find that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n (s)_{n+1}}{(n+2)!} \Phi(x,\mathrm{s}+\mathrm{n}+1,a)$$

$$= \left\{\frac{1}{x}+1\right\}\Phi(x,s,a) - \frac{a^{-s}}{x}\} + \frac{2}{s-1}\{\left(\frac{1}{x}-1\right)\Phi(x,\mathrm{s}-1,a) - \frac{a^{-s+1}}{x}\}$$

$$= \left\{\frac{1}{x}+1\right\}\Phi(x,s,a) + \frac{2}{s-1}\left(\frac{1}{x}-1\right)\Phi(x,\mathrm{s}-1,a) - \frac{a^{-s}}{x}\left(1+\frac{2a}{s-1}\right) \tag{3.8}$$

The relation between (3.8) easily the desired result (3.1) provided that the series converges.

Proof of (3.2): Replacing the summation n by n+1 in (3.3), and setting $\lambda = s-1$, we have

$$\sum_{n=0}^{\infty} \frac{(s)_n}{(n+1)!} \Phi(x,\mathrm{s}+\mathrm{n},a)t^{n+1}$$

$$= \frac{1}{s-1}\{\Phi(x,\mathrm{s}-1,a-t) - \Phi(x,\mathrm{s}-1,a)\},\ |t| < |a| \tag{3.9}$$

Now taking $t = -1$ in(3.9) and using (3.7), therein, we have easily get the desired result (3.2).

4. Special Cases

If we set $x = 1$ in (3.1), we immediately obtain a known result [7, p., 49, eq 2.6], which reduces to another known summation formula by Singh and Verma [5] when we put $a = 2$ therein.

Also, the result (3.2) contains the known result recently established by [8, p. 131 eq. 2.4].

It may be remarked here that on taking $x = 1, a = 2$ in summation formula(3.3), we readily get,

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\{\zeta(s+n)-1\}t^n = \zeta(\lambda, 2-t) \quad ,|t| < 2, \lambda \neq 1 \tag{4.1}$$

Or equivalently, that

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \zeta(s+n) t^n = \zeta(\lambda, 1-t) \quad , |t| < 2, \lambda \neq 1 \tag{4.2}$$

(4.2) Provides interesting unfications of the summation formulas like (2.3) and

$$\sum_{n=2}^{\infty} (-1)^n f(\zeta(n)) = 1/2 \tag{4.3}$$

$$\sum_{n=1}^{\infty} f(\zeta(2n)) = 3/4 \text{ etc.} \tag{4.4}$$

Indeed a large number of summation formulas have been evaluated from (4.2)(see e.g.[8] and [9]).

Conclusion: The Riemann zeta functions are very important function and they attract the researcher. In this paper, we establish three general multiple-series identities involving a general class of polynomials and we also deduct some useful results out of which some are known and some are new. The summation formula is also found using the three series.

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References:

1. Choi, J., Srivastava H.M., (1997), Sums associated with the zeta function, J. Math. Anal, Appl. 206, 103-120.
2. Erdelyi, A., et.al.(1953) ,Higher Transendental Functions, MeGraw Hill, New York, Toronto and London, 1.
3. Ivic A(1980)., The Riemann zeta Function, John wiley and Sons, Neww York,
4. Shallit J.D., Zikan K.,(1986), A theorem of goldback, Amer math Monthly,93, 402,403.
5. Singh R.J., Verma D.P.,(1983), Some series involving Riemann zeta function , Yokahama Math J., 31,1-4.

6. Srivastava H.M., A class of Hypergeometric polynomials(1987) , J. Austral Math. Soc., 43(ser. A)187-198.
7. Srivastava H.M.,,(1987), Some infinite series associated with Riemann zeta functions, Yokahama Math. J., 35, 47-50.
8. Srivastava H.M.,,(1988), Sums of certain series of Riemann zeta functions, J. Math.,Anal.,Appli.,134(1), 129-140.
9. Srivastava H.M.,,(1988), A unified presentain of certain classes of Riemann zeta functions, Riv. Mat. Univ. Parma 4,1-23.
10. Sitchmarch E.C.(1951), The theory of Riemann zeta Functions, Clarendon press oxford and London.